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A MEASURING ARGUMENT FOR FINITE PERMUTATION GROUPS

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ABSTRACT

Let G be a finite transitive permutation group on a finite set S. Let A be a nonempty subset of S and denote the pointwise stabilizer of A in G by $C_G(A)$. Our main result is the following inequality: $[G : C_G(A)] \geq |G|^{|A|/|S|}$.

I. Introduction

Let G be a finite transitive group of permutations of a finite set S of size n. If A is a nonempty subset of S of size k, let $C_G(A)$ denote the pointwise stabilizer of A in G. It is well known that $[G : C_G(A)] \leq n(n-1)\cdots(n-k+1)$. However, as far as we know, no lower bound for $[G : C_G(A)]$ had been described in the literature. The main purpose of this paper is to prove that

$$[G : C_G(A)] \geq |G|^{|A|/|S|}.$$

The methods of this paper are variations of those used by A. Chermak and A. Delgado in their paper *A measuring argument for finite groups* [1]. While their paper dealt with finite groups G acting on a finite group H, we expand the discussion to the case of finite groups G acting on a finite set S.

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II. The basic definitions and results

Let G be a finite group acting on a finite set S and let $\mathcal{A}(S)$ denote the set of all nonempty subsets of S. If $A \in \mathcal{A}(S)$, then $C_G(A)$ denotes the pointwise stabilizer of A, which will be referred to as a **stabilizer** of G on S. Finally, let $\alpha > 1$ be a real number.

Definition 1: We define:

$$m_\alpha = m_\alpha(G,S) = \max\{\alpha^{|A|}|C_G(A)| \mid A \in \mathcal{A}(S)\};$$
$$\alpha M = \alpha M(G,S) = \{A \in \mathcal{A}(S) \mid \alpha^{|A|}|C_G(A)| = m_\alpha\};$$
$$\alpha M^* = \text{ the set of maximal members of } \alpha M, \text{ under inclusion;}$$
$$\alpha M_* = \text{ the set of minimal members of } \alpha M, \text{ under inclusion.}$$

Our first observation is

LEMMA 1: *The subsets αM, αM^* and αM_* of $\mathcal{A}(S)$ are closed under the action of G.*

Proof: Obvious. ∎

The following lemma contains some basic properties of αM. In this lemma we use the notation $C_G(\emptyset) = G$.

LEMMA 2: *Let $A, B \in \alpha M$ and suppose that either $A \cap B \neq \emptyset$ or $m_\alpha \geq |G|$. Then the following statements hold:*

(1) $A \cup B \in \alpha M$;
(2) $C_G(A \cap B) = C_G(A)C_G(B)$;
(3) *if* $A \cap B \neq \emptyset$, *then* $A \cap B \in \alpha M$;
(4) *if* $A \cap B = \emptyset$, *then* $m_\alpha = |G|$.

Proof: Since $A \in \alpha M$, we have

$$\alpha^{|A|}|C_G(A)| \geq \alpha^{|A\cup B|}|C_G(A\cup B)| = \frac{\alpha^{|A|}\alpha^{|B|}}{\alpha^{|A\cap B|}}|C_G(A\cup B)| \tag{2-1}$$

which, after reduction, yields

$$\frac{\alpha^{|B|}}{\alpha^{|A\cap B|}} \leq \frac{|C_G(A)|}{|C_G(A)\cap C_G(B)|} = \frac{|C_G(A)C_G(B)|}{|C_G(B)|} \leq \frac{|C_G(A\cap B)|}{|C_G(B)|} \tag{2-2}$$

with $|C_G(A\cap B)| = |G|$ if $A \cap B = \emptyset$.

Suppose, first, that $A \cap B \neq \emptyset$. Then it follows from (2-2) that $\alpha^{|B|}|C_G(B)| \leq \alpha^{|A\cap B|}|C_G(A\cap B)|$, and since $B \in \alpha M$, the equality must hold. This implies that $A \cap B \in \alpha M$ and equalities holds throughout the expressions (2-1) and (2-2). In particular, as $A \in \alpha M$, the equality in (2-1) implies that $A \cup B \in \alpha M$ and the equalities in (2-2) imply that $C_G(A \cap B) = C_G(A)C_G(B)$ holds. We have shown that (1), (2) and (3) hold in this case, as claimed.

Suppose, now, that $A \cap B = \emptyset$. In this case $\alpha^{|B|}|C_G(B)| = m_\alpha \geq |G|$ and (2-2) implies $\alpha^{|B|}|C_G(B)| \leq |G|$. Hence $m_\alpha = |G|$ and consequently equalities hold throughout the expressions (2-1) and (2-2). In particular, $A \cup B \in \alpha M$ and $G = C_G(A)C_G(B)$ hold. We have shown that (1), (2) and (4) hold in this case, as claimed.

The proof of the lemma is complete. ∎

Lemmas 1 and 2 yield the following results about the subsets αM^* and αM_* of αM.

LEMMA 3: *If $A \in \alpha M^* \cup \alpha M_*$, then A is a block for G on S.*

Proof: Since $A \in \alpha M$, $A^g \in \alpha M$ for every $g \in G$. If $A \cap A^g \neq \emptyset$ for some $g \in G$, then by Lemma 2, $A \cup A^g \in \alpha M$ and $A \cap A^g \in \alpha M$. But $A \cap A^g \subseteq A \subseteq A \cup A^g$ and $A \in \alpha M^* \cup \alpha M_*$, hence either $A = A \cup A^g$ or $A = A \cap A^g$. Consequently $A = A^g$, which implies that A is a block. ∎

LEMMA 4:

(1) *If $m_\alpha \geq |G|$, then $\alpha M^* = \{U\}$, where $U =_{\text{def}} \cup \alpha M$. Moreover, $U^g = U$ for all $g \in G$ and $C_G(U) \lhd G$.*

(2) *If $m_\alpha > |G|$, then $\alpha M_* = \{I\}$, where $I =_{\text{def}} \cap \alpha M$. Moreover, $I^g = I$ for all $g \in G$ and $C_G(I) \lhd G$.*

Proof: If $m_\alpha \geq |G|$, then it follows by Lemma 2(1) that $U \in \alpha M$ and consequently $\alpha M^* = \{U\}$. If $g \in G$, then by Lemma 1, $U^g = U$ and hence $(C_G(U))^g = C_G(U^g) = C_G(U)$. The proof of (1) is complete.

If $m_\alpha > |G|$, then by Lemma 2(4), $A \cap B \neq \emptyset$ for all $A, B \in \alpha M$ and consequently, by Lemma 2(3), $I \in \alpha M$. Hence $\alpha M_* = \{I\}$ and by Lemma 1, $I^g = I$ for all $g \in G$. This implies that $(C_G(I))^g = C_G(I^g) = C_G(I)$ for all $g \in G$, and the proof of (2) is complete. ∎

Remark: Clearly we can choose α so that either of the conditions in Lemma 4 holds. In particular, if $\alpha \geq |G|^{1/|S|}$, then $\alpha^{|S|} \geq |G|$ and we get $m_\alpha \geq \alpha^{|S|}|C_G(S)| \geq |G|$. Similarly, $\alpha > |G|^{1/|S|}$ implies that $m_\alpha > |G|$.

We now turn our attention to finite transitive permutation groups G, acting on finite sets S satisfying $\alpha^{|S|} \geq |G|$. We denote by $\mathcal{T}$ the set of triples (G, S, α)

satisfying these conditions. In this case $C_G(S) = 1$ and $m_\alpha \geq \alpha^{|S|} \geq |G|$. We shall also use the notation U and I introduced in Lemma 4.

LEMMA 5: *Let $(G, S, \alpha) \in \mathcal{T}$. Then $S \in \alpha M$ and $m_\alpha = \alpha^{|S|} \geq |G|$. If $\alpha^{|S|} > |G|$, then $\alpha M = \{S\}$ and $m_\alpha = \alpha^{|S|} > |G|$.*

Proof: Since $m_\alpha \geq |G|$, it follows by Lemma 4(1) that U is closed under the action of G. But G is transitive on S, so $U = S \in \alpha M$ and $m_\alpha = \alpha^{|S|} \geq |G|$. If $\alpha^S > |G|$, similar considerations yield, in view of Lemma 4(2), that $I = S$, which implies that $\alpha M = \{S\}$ and $m_\alpha = \alpha^{|S|} > |G|$. ∎

LEMMA 6: *Let $(G, S, \alpha) \in \mathcal{T}$. Then the action of G on αM_* is transitive. In particular, if $A, B \in \alpha M_*$, then $|A| = |B|$.*

Proof: Let $A, B \in \alpha M_*$. By Lemma 1 it suffices to prove that $B = A^g$ for some $g \in G$. Let $a \in A$ and $b \in B$. Since G is transitive on S, there exists $g \in G$ such that $b = a^g$. By Lemma 1, $A^g \in \alpha M_*$ and since $b \in B \cap A^g$, it follows by Lemma 2(3) that $B \cap A^g \in \alpha M$. But B and A^g are minimal in αM under inclusion, so $B = A^g$, as required. ∎

We end this section with a result dealing with a finite simple permutation group, not necessarily transitive, acting on a nonempty finite set S. It turns out that in this case the results of Lemma 5 also hold. We still use the notation of Lemma 4.

LEMMA 7: *Let G be a finite simple permutation group on a finite set S and suppose that $\alpha^{|S|} \geq |G|$. Then $S \in \alpha M$ and $m_\alpha = \alpha^{|S|} \geq |G|$. If $\alpha^{|S|} > |G|$, then $\alpha M = \{S\}$ and $m_\alpha = \alpha^{|S|} > |G|$.*

Proof: Since $m_\alpha \geq |G|$, it follows by Lemma 4(1) that $\alpha M^* = \{U\}$ and $C_G(U) \lhd G$. Since $U \neq \emptyset$, $C_G(U) \neq G$, and the simplicity of G forces $C_G(U) = 1$. Hence $\alpha^{|S|} \leq m_\alpha = \alpha^{|U|}$, yielding $S = U \in \alpha M$ and $m_\alpha = \alpha^{|S|} \geq |G|$. In the case when $\alpha^{|S|} > |G|$, Lemma 4(2) similarly yields $I = U = S$, which implies $\alpha M = \{S\}$ and $m_\alpha = \alpha^{|S|} > |G|$. ∎

III. Dominated stabilizers

In this section we define and investigate finite permutation groups with "dominated stabilizers". From these investigations the main results of this paper will emerge.

Definition 2: Let G be a finite permutation group of the nonempty finite set S. We say that G has **dominated stabilizers** on S if for each $A \subseteq S$, the following inequality holds:

$$|C_G(A)| \leq |G|^{1-|A|/|S|}.$$

If for each nonempty proper subset A of S we have $|C_G(A)| < |G|^{1-|A|/|S|}$, then we say that G has **strictly dominated stabilizers** on S.

Concerning G, S and α we shall also use the assumptions and the notation of Definition 1. We start with

PROPOSITION 8: *Let G be a finite permutation group of the nonempty finite set S and suppose that $\alpha^{|S|} = |G|$. Then*

(1) *G has dominated stabilizers on S if and only if $S \in \alpha M$;*
(2) *G has strictly dominated stabilizers on S if and only if $\alpha M = \{S\}$.*

Proof: If $S \in \alpha M$, then $m_\alpha = \alpha^{|S|}$, so for each $A \subseteq S$ we have $\alpha^{|A|}|C_G(A)| \leq \alpha^{|S|}$. Hence

$$|C_G(A)| \leq \alpha^{|S|-|A|} = (|G|^{1/|S|})^{|S|-|A|} = |G|^{1-|A|/|S|},$$

which implies that G has dominated stabilizers on S. If, moreover, $\alpha M = \{S\}$, then for each nonempty proper subset A of S we have $|C_G(A)| < |G|^{1-|A|/|S|}$, implying that G has strictly dominated stabilizers on S.

Conversely, if G has dominated stabilizers on S, then for each $A \subseteq S$ we have

$$|C_G(A)| \leq |G|^{1-|A|/|S|} = (|G|^{1/|S|})^{|S|-|A|} = \alpha^{|S|-|A|},$$

whence $\alpha^{|A|}|C_G(A)| \leq \alpha^{|S|}$ and it follows that $S \in \alpha M$. Similarly, if G has strictly dominated stabilizers, then for each nonempty proper subset A of S we have $\alpha^{|A|}|C_G(A)| < \alpha^{|S|}$, which implies that $\alpha M = \{S\}$. ∎

Our main result in this paper is

THEOREM 9: *Let G be a finite permutation group on a finite set S. If $A \subseteq S$ and G is either transitive on S or a simple group, then the following inequality holds:*

$$[G : C_G(A)] \geq |G|^{|A|/|S|}. \tag{9-1}$$

Proof: Let $\alpha = |G|^{1/|S|}$. If G is transitive on S, then, by Lemma 5, $S \in \alpha M$ and by Proposition 8 $|C_G(A)| \leq |G|^{1-|A|/|S|}$, which implies the inequality (9-1).

Similarly, if G is a simple group, then, by Lemma 7, $S \in \alpha M$ and again Proposition 8 yields the required inequality. ∎

We conclude this paper with the following result dealing with simple transitive permutation groups.

COROLLARY 10: *Let G be a finite simple transitive permutation group on a finite set S. Then G has strictly dominated stabilizers on S.*

Proof: Since G is simple, $|S| \geq 2$. By Theorem 9, G has dominated stabilizers on S. Let $\alpha = |G|^{1/|S|}$. Then by Lemma 5, $S \in \alpha M$ and by Lemma 6, $T =_{\text{def}} \alpha M_*$ consists of subsets of S of equal size. If T consists of sets of size 1, then $\{a\} \in T$ for some $a \in S$ and $m_\alpha = \alpha|C_G(a)| = \alpha^{|S|} = |G|$. Hence $|S| = [G : C_G(a)] = \alpha$ and $|G| = \alpha^{|S|} = |S|^{|S|}$, a contradiction, since $|G| \leq |S|!$. Thus T consists of sets of size 2 at least.

By Proposition 8(2) it suffices to show that $\alpha M = \{S\}$, or equivalently, that $S \in T$. If $|T| = 1$ and $A \in T$, then, by Lemma 1, A is closed under the action of G and it follows by the transitivity of G that $A = S \in T$, as required. So we may assume that $|T| > 1$.

We proceed by induction on S. If $|S| = 2$, then $S \in \alpha M$, and since αM contains no sets of size 1, as shown above, it follows that $\alpha M = \{S\}$, as required. So suppose that $|S| = k > 2$ and the corollary holds for all S with $|S| < k$. We may assume that $S \notin T$. By Lemma 1, $T = \{A^g |\, g \in G\}$ for some $A \subset S$ and we may assume that $1 < |A| < |S|$. Moreover, by Lemma 2(3), if $A, B \in T$ are distinct and $A \cap B \neq \emptyset$, then $A \cap B \in \alpha M$, which is a contradiction, since $T = \alpha M_*$. Since G is transitive on S, it follows that S is a disjoint union of elements of T. Hence $1 < |T| = |S|/|A| < |S|$. Consider the transitive action of G on T and let $N = C_G(T)$. Then $N \trianglelefteq G$. If $N = G$, then it follows by the transitivity of G on S that $A = S \in T$, as required. So $N = 1$ and G is a transitive permutation group of T. Since $|T| < |S|$, it follows by induction that G has strictly dominated stabilizers on T. Thus, as $|T| > 1$, it follows that

$$|C_G(A)| \leq |N_G(A)| < |G|^{1-1/|T|} = |G|^{1-|A|/|S|},$$

where $N_G(A)$ denotes the pointwise stabilizer of A in the action of G on T, which is equal to the stabilizer of A in the action of G on S. But $A, S \in \alpha M$, so $|C_G(A)| = |G|^{1-|A|/|S|}$, a contradiction. This completes the proof of the corollary. ∎

References

[1] A. Chermak and A. Delgado, *A measuring argument for finite groups*, Proceedings of the American Mathematical Society **107** (1989), 907–914.